

Exercise: (Knaster-Tarski Fixed Point Theorem)

Let $L = (D, \leq)$ be a complete lattice and let

$f: D \rightarrow D$ be a monotone function with respect to $\leq$, i.e.

if $a \leq b$ then $f(a) \leq f(b)$.

Then the set of fixed points of f forms a complete lattice under $\leq$.

And in particular,

f has a least fixed point. ★

Proof: Fill in proof skeleton on next page for the claim ★. (You prove only part of Knaster-Tarski.)

Couple definitions

Partial Order $(L, \leq)$ satisfying:

- 1) $\forall x, y \in L. x \leq y \wedge y \leq x \rightarrow x = y$
- 2) $\forall x. x \leq x$
- 3) $\forall x, y, z. x \leq y \wedge y \leq z \rightarrow x \leq z$

Complete Lattice $(L, \leq)$ satisfying:

- 1) $(L, \leq)$ is a partial order and
- 2) for every $X \subseteq L$, there exists a **greatest lower bound**
I.e., an element $\bigcap X \in L$ s.t.
 - a) $\bigcap X \leq x$ for every $x \in X$
(**lower bound of X**) and
 - b) $y \leq \bigcap X$ for every y s.t. $y \leq x$ for every $x \in X$
(**greatest among all lower bounds**)

Let $\text{Pre} := \{ x \in D : f(x) \leq x \}$.

Note: Pre is the set of "pre-fixed points" of f .

Let $y := \bigcap \text{Pre}$.

Argue that y is a fixed point of f . Show it is a pre-fixed point:

$$f(y) \leq y$$

And show it's a "post-fixed point":

$$y \leq f(y).$$

Conclude $y = f(y)$, i.e. y is a fixed point.

Fill.

Next, argue that for any fixed
point x , $y \leq x$.

Fill.