

Propositional Logic

Proposition : statement that is either true or false.

Example : Boulder is the capital of Colorado.

Syntax : Propositions $p_1, p_2, p_3, \dots \in P$
 $\varphi ::= p \in P \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg \varphi$

Valuation (equivalently, "Assignment")

$$V : P \rightarrow \{0, 1\}$$

Semantics : $v \models \varphi$

Derived:

$$\varphi \Rightarrow \psi := \neg \varphi \vee \psi$$

$$\top := \varphi \vee \neg \varphi$$

$$\perp := \neg \top$$

$$v \models p \quad \text{iff} \quad v(p) = 1$$

$$v \models \varphi \wedge \varphi' \quad \text{iff} \quad v \models \varphi \text{ and } v \models \varphi'$$

$$v \models \varphi \vee \varphi' \quad \text{iff} \quad v \models \varphi \text{ or } v \models \varphi'$$

$$v \models \neg \varphi \quad \text{iff} \quad v \not\models \varphi$$

Example :

$P := \text{"Boulder is the capital of Colorado"}$

$$v(P) = 0$$

$$v \models \neg P$$

$$\Leftrightarrow v \not\models P \Leftrightarrow v(P) = 0 \neq 1$$

Satisfiable : φ is sat if $v \models \varphi$
for some $v : P \rightarrow \{0, 1\}$.

Valid : φ is valid if $v \models \varphi$
for every $v : P \rightarrow \{0, 1\}$.

Relevance Lemma : Let $X \subseteq P$ be the
propositions appearing in φ . If v and v'
agree on X then $v \models \varphi$ iff $v' \models \varphi$.

Proof : By induction on syntax of φ .

Notation: Γ and Δ denote sets of formulas, possibly infinite.

" $v \models \Gamma$ " means $v \models \varphi$ for every $\varphi \in \Gamma$.

" $\Gamma \models \varphi$ " means for every $v : P \rightarrow \{0,1\}$ s.t. $v \models \Gamma$ we have $v \models \varphi$.

A proof system for reasoning in propositional logic

Goal: Given assumptions Γ and conclusion φ , prove $\Gamma \models \varphi$.

How could we prove $\Gamma \models \varphi$? There may be an infinite number of assumptions in Γ , and an infinite number of ways to satisfy them, and for each satisfying v , we need to show that $v \models \varphi$.

Idea: Design a "mechanical" procedure which is conservative in its answers, i.e. can never be caught giving a wrong answer. If the procedure claims $\Gamma \models \varphi$ then it's true.

Resolution: proofs by refutation

Given: assumptions Γ and conclusion φ .

Observation: $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg \varphi\}$ is unsat.

Build the proof by proving that a set of clauses is unsat.

Conjunctive Normal Form (CNF)

"Literal": proposition or its negation, e.g. $P, \neg P$

"Clause": disjunction of literals, e.g. $p \vee q \vee \neg r$

represent clauses as sets: $\{P, q, \neg r\}$

"Empty Clause": $\emptyset$, equivalent to $\perp$

"CNF formula": conjunction of clauses

represent as set of sets of literals

Theorem: Every φ is equivalent to some φ' in CNF.

Example: $p \wedge q \vee t \wedge r =: \varphi$

$$(p \vee (t \wedge r)) \wedge (q \vee (t \wedge r))$$

$$(p \vee t) \wedge (p \vee r) \wedge (q \vee t) \wedge (q \vee r) =: \varphi'$$

Represent φ' as: $\{\{p, t\}, \{p, r\}, \{q, t\}, \{q, r\}\}$

Note: Size of φ' is polynomially-bounded in size of φ .

Really??

$$p_1 \wedge q_1 \vee p_2 \wedge q_2 \vee \dots \vee p_n \wedge q_n =: \varphi$$

Distributing leads to φ' exponentially-larger than φ .

Tseitin technique produces equisatisfiable φ' of linear size.

"Refutation" for Γ (think: $\Gamma := \Gamma' \cup \{\neg\varphi\}$)

Sequence $C_1, C_2, \dots, C_n$ s.t. $\forall i \in [n]$,

C_i is either:

1) an assumption in Γ

2) $\exists j, k < i$ s.t. C_i is resolvent of C_j, C_k .

and $C_n = \emptyset$

Resolution:
$$\frac{C \cup \{p\} \quad D \cup \{\neg p\}}{C \cup D} \quad \leftarrow \text{resolvent w.r.t. } p \text{ of } C_j, C_k$$

Example: $\Gamma = \{ \{p\}, \{q, \neg p\}, \{\neg p, \neg q\} \}$

$$C_1 = \{p\}$$

$$C_2 = \{q, \neg p\}$$

$$C_3 = \{\neg p, \neg q\}$$

$$C_4 = \{q\}$$

$$C_5 = \{\neg p\}$$

$$C_6 = \emptyset$$

Soundness: If there exists a refutation for Γ then Γ is unsatisfiable.

Proof: Exercise

Completeness: If Γ is unsat there exists a refutation for it.

Proof: Homework

Compactness of propositional logic

Theorem Γ is satisfiable iff Γ is finitely-satisfiable
(Compactness)

Γ is satisfiable if there is some $v: P \rightarrow \{0,1\}$
s.t. $v \models \varphi$ for every $\varphi \in \Gamma$.

Γ finitely-satisfiable: every finite subset
of Γ is satisfiable

Proof ($\Rightarrow$) Trivial

($\Leftarrow$) Let $p_1, p_2, p_3, \dots$ be enumeration of P .

Assume Γ is finitely-sat.

Define $\Delta_0 := \Gamma$.

$$\Delta_{i+1} := \begin{cases} \Delta_i \cup \{p_i\} & \text{if } \Delta_i \cup \{p_i\} \text{ is finitely-sat} \\ \Delta_i \cup \{\neg p_i\} & \text{o.w.} \end{cases}$$

Claim $\forall i$ Δ_i is finitely-sat

Prove by induction.

Base $\Delta_0 = \Gamma$, which is finitely-sat.

Induction step $\Delta_{i+1} = \Delta_i \cup \{p_i\}$, by defn.

o.w. $\Delta_{i+1} = \Delta_i \cup \{\neg p_i\}$

Aside: Let X be any set of formulas
and let φ be any formula.

Lemma: if X is finitely-sat then

either $X \cup \{\varphi\}$ or $X \cup \{\neg \varphi\}$ is finitely-sat.

Proof Assume for contradiction that neither is.

Let U, V be ^{finite} subsets of $X \cup \{\varphi\}$
and $X \cup \{\neg \varphi\}$ which are unsat.

φ not in X and $\neg \varphi$ not in X ,

o.w. contradiction.

Have: $\varphi \in U$ and $\neg \varphi \in V$ (o.w. sat)

Let v be such that $v \models U \setminus \{\varphi\} \cup$
 $V \setminus \{\neg \varphi\}$

Suppose wlog that $v \models \varphi$,

then $v \models U$, a contradiction.

Define $\Delta := \bigcup_{i \in \mathbb{N}} \Delta_i$.

Claim Δ is finitely-sat. [Aside: $\Delta_i \subseteq \Delta_{i+1} \forall i$]

Proof

Suppose $X \in \mathcal{P}^{fm}(\Delta)$. $\exists j$ s.t. $X \subseteq \Delta_j$.

X is sat because Δ_j is finitely-sat.

Define $v(p_i) = \begin{cases} 1 & \text{if } p_i \in \Delta \\ 0 & \text{o.w.} \end{cases}$

[Check: is v well defined?]

Claim: $v \models \Gamma$.

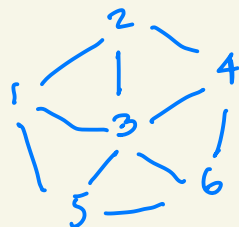
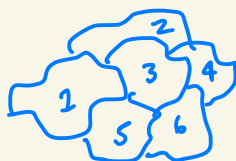
Proof Let $\varphi \in \Gamma$. Let Y be the propositions appearing in φ . Consider $Z := \{\varphi\} \cup (\{y, \neg y : y \in Y\} \cap \Delta)$

Since Z is finite and Δ is finitely-sat, let v' be s.t. $v' \models Z$, and in particular $v' \models \varphi$. But v and v' agree on Y and thus by relevance lemma $v \models \varphi$.

Application to 4-coloring infinite planar graphs.

Theorem (Haken-Appel) Every finite planar* graph can be colored with 4 colors.

Territorial maps



Planar graph

graph drawable on the plane s.t.
no two edges cross.

Graph $G = (V, E)$. $E \subseteq V \times V$. Irreflexive, E is symmetric.

K-coloring $c: V \rightarrow \{1, \dots, K\}$ s.t. $uv \in E \Rightarrow c(u) \neq c(v)$

Claim Every planar (possibly infinite) graph is 4-colorable.

Proof

$$\bigwedge_{i \in V} \bigwedge_{j \neq i \in [K]} (\neg x_{ij} \vee \neg x_{ji})$$

$$\bigwedge_{i \in V} \bigvee_{j \in [K]} x_{ij}$$

$$\bigwedge_{i, i' \in V} \bigwedge_{j \in [K]} (\neg x_{ij} \vee \neg x_{i'j}) =: \prod_{G, K}$$

adjacent vertices have different colors

$\Gamma_{G, 4}$ is satisfiable by Haken-Appel,

for any finite planar G .

Suppose G^* is infinite and planar.

$\Gamma_{G^*, 4}$ is satisfiable by compactness,

because every subset that is finite is.

[check: certainly each subset corresponding to 4-colorability of a finite subgraph of G^* is satisfiable. But what about all the other subsets?]